E9 203 Compressed sensing and sparse SP

8 Mar. 2021.

Last time:

Minimal # meas. to recover all s-sparse vecs $x \in \mathbb{C}^N$ from $y = Ax \in \mathbb{C}^m$. $\boxed{m \geq 2s}$ is necessary and sufficient.

Today:

Recovery of individual sparse vecs

NP hardness of ℓ_0-min.

Basic algos.

Recovery of individual sparse vecs:

Thm. For any $N \geq s+1$, given an s-sparse $x \in \mathbb{C}^N$, $\exists$ a measurement matrix $A \in \mathbb{C}^{m \times N}$ with $m = s+1$ s.t. x can be reconstructed from $y = Ax$ as a soln of

$$\boxed{(P_0): \min_{x \in \mathbb{C}^N} \|x\|_0 \quad \text{s.t.} \quad Ax = y.}$$

Proof: Suppose $A \in \mathbb{C}^{(s+1) \times N}$ is s.t. s-sparse x cannot be recovered from $y = Ax$ via (P_0).

$\Rightarrow \exists z \in \mathbb{C}^N$, $z \neq x$, with $S \triangleq \mathrm{supp}(z) = \{j_1, \dots, j_s\}$ of size at most s (if $\|z\|_0 < s$, fill arbitrary indices whose entries are 0 into S), s.t. $Az = Ax$.

If $\mathrm{supp}(x) \subset S$,

$$A(z - x) = 0 \Rightarrow A_S \overbrace{(z-x)_S}^{\text{nonzero.}} = 0 \qquad A_S : s \text{ cols. } s+1 \text{ rows.}$$

$\Rightarrow \boxed{A_{[s],S}}$ is not invertible, size $s \times s$

$\Rightarrow f_1(A) \triangleq \det(A_{[s],S}) = 0.$

If $\mathrm{supp}(x) \not\subset S$, then define the subspace

$$V \triangleq \{\underbrace{u \in \mathbb{C}^N : \mathrm{supp}(u) \subset S}_{\dim s}\} + \underbrace{\mathbb{C}x}_{\dim 1}. \qquad z, x \in V \quad z \neq x$$

V has dimension $s+1$.

The mapping $G: V \longrightarrow \mathbb{C}^{s+1} : v \longrightarrow Av$ is not invertible $\because G(z - x) = A(z-x) = 0$, $z \neq x$.

The map G, in the basis $(\underbrace{e_{j_1}, e_{j_2}, \dots, e_{j_s}}_{\text{cols of } I_N \text{ indexed by } j_1 \cdots j_s}, x)$ has the form

$$B_{x,S} \triangleq \begin{bmatrix} a_{1,j_1} & a_{1,j_2} & \cdots & a_{1,j_s} & \sum_{j \in \mathrm{supp}(x)} x_j a_{1,j} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{s+1,j_1} & a_{s+1,j_2} & & a_{s+1,j_s} & \underbrace{\sum_{j \in \mathrm{supp}(x)} x_j a_{s+1,j}}_{\text{l.c. of cols of } A} \end{bmatrix}$$

$\Rightarrow g_S(A) = \det(B_{x,S}) = 0.$

Thus, if x is not recoverable from $y = Ax$ via (P_0), then A satisfies

$$A \in f_1^{-1}(\{0\}) \cup \Big\{ \bigcup_{|S|=s} g_s^{-1}(\{0\}) \Big\}$$

Thus, if $A \notin \Big[f_1^{-1}(\{0\}) \cup \Big\{ \bigcup_{|S|=s} g_s^{-1}(\{0\}) \Big\} \Big]$,

then x is recoverable from $y = Ax$ via ℓ_0 min.

But f_1, and all g_s, $|S| = s$ are polynomial fns. of the entries of $A \Rightarrow$ the sets $f^{-1}(\{0\})$ and $g_s^{-1}(\{0\})$, $|S| = s$ have Lebesgue measure 0, and so does their union. Hence, choosing entries of A outside of this union of measure 0 ensures that x can be recovered from $y = Ax$ via ℓ_0 min. ☒

NP hardness of ℓ_0 minimization

$$(P_0) \quad \min_{z \in \mathbb{C}^N} \|z\|_0 \quad \text{s.t.} \quad y = Az$$

is hard: if we know that the optimal z is s-sparse, straightforward approach:

$$A_S u = y \quad \forall \; S \subset [N], |S| = s.$$

$$A_S^H A_S u = A_S^H y \quad \text{square system.}$$

Each system is solvable w/ poly(s^3) complexity.

But, # subsets $= \binom{N}{s}$ too large.

If $N = 1000$, $s = 10$. $\binom{1000}{10} \geq \left(\frac{1000}{10}\right)^{10} = 10^{20}$

systems of size 10×10.

If each system takes 1ns. Need 10^{11} secs $\approx$ 3000 years!

Can s.t. (P_0) is in fact intractable by any approach, not just the approach above.

Thm. 2.17 For any $\eta \geq 0$, the ℓ_0 min. problem

$$(P_{0,\eta}): \min_{z \in \mathbb{C}^N} \|z\|_0 \quad \text{s.t.} \quad \|Az - y\|_2 \leq \eta$$

for general $A \in \mathbb{C}^{m \times N}$ and $y \in \mathbb{C}^m$ is NP hard.

$\{1,2,3\}$ $\{4,5,6\}$

$\{6,7,8\}$ $\{3,20,8\}$ … 1000

$N = 1000$, $m = 20$

Exact cover by 3-sets:

Given a collection $\{C_i, i \in [N]\}$ of 3-element subsets of $[m]$, does $\exists$ an exact cover (partition) of $[m]$. That is, a set $J \subset [N]$ s.t.

$$\bigcup_{j \in J} C_j = [m] \quad \text{and} \quad C_j \cap C_{j'} = \emptyset \;\; \forall j \neq j', \; j, j' \in J \;?$$

NP hard.

Chapter 3: Basic algorithms

Optimization methods:

General opt. pb.

$$\left\{ \min_{x \in \mathbb{R}^N} \underbrace{F_0(x)}_{\text{objective fn.}} \quad \text{s.t.} \quad \underbrace{F_i(x)}_{\text{constraint fns.}} \leq b_i, \; i \in [n] \right\}$$

$G(x) = c$, equiv. to $G(x) \leq c$ and $-G(x) \leq -c$.

Encompasses ...

If $F_0, F_1, \ldots, F_n$ convex $\Rightarrow$ convex opt. pb.

Our problem:

$$\min_{z \in \mathbb{R}^N} \|z\|_0 \quad \text{s.t.} \quad \underline{Az = y}$$

is nonconvex, NP hard in general.

Since $\|x\|_q^q \longrightarrow \|x\|_0$ as $q \to 0^+$, can approximate (P_0) by

$$(P_q) \quad \min_{z \in \mathbb{R}^N} \|z\|_q \quad \text{s.t.} \quad Az = y$$

$q > 1$, even 1-sparse vecs are not solns. of (P_q). (Ex. 3.1)

For $0 < q < 1$, (P_q) is nonconvex, NP hard in gen.

$q = 1$: We get the convex problem:

$$\boxed{(P_1) \quad \min_{z \in \mathbb{R}^N} \|z\|_1 \quad \text{s.t.} \quad Az = y.}$$

- ℓ_1 min, basis pursuit
- "convex relaxation" of (P_0) (See App. B3).

<u>Thm.</u> 3.1 [ℓ_1 min $\Rightarrow$ sparse solns!]

Let $A \in \mathbb{R}^{m \times N}$ be a meas. matrix with cols. $a_1, \ldots, a_N$. Assuming <u>uniqueness</u> of a minimizer $x^\#$ of (P_1), the system $\underline{\{a_j, j \in \operatorname{supp}(x^\#)\}}$ is lin. indep., and $\underline{\|x^\#\|_0 \leq m}$.